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Special Cubics And Quartics Produced
By Projective Pencils Of Straight Lines
And Conics.

SPECIAL CUBICS AND QUARTICS PRODUCED BY
PROJECTIVE PENCILS OF STRAIGHT
LINES AND CONICS

BY

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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPER-
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SPECIAL CUBICS AND QUARTICS PRODUCED BY PROJECTIVE PENCILS

Introduction

The purpose of the following is to determine special cubics and quartics found by taking the product of two projective pencils of conics, or the product of a projective pencil of lines and a corresponding pencil of conics, or the product of a pencil of cubics and a pencil of lines. (Examples of each case will be given in Part II). It will also be shown how such curves may be constructed geometrically.

Definition. An equation of the form

$$U + \lambda V = 0 \text{ ----- (1)}$$

where $U=0$ and $V=0$ are two equations in x and y will be called a pencil. That is for each value of λ , (1) represents one element of the pencil.

Definition of Product of two Projective Pencils.

Consider the two pencils

$$\begin{aligned} U + \lambda V &= 0, \\ R + \lambda S &= 0. \end{aligned} \text{ ----- (2)}$$

These are said to be two projective pencils. If λ is eliminated from (2) an equation of the form

$$US - RV = 0 \text{ ----- (3)}$$

is obtained. Equation (3) is called the product of (2).

We shall now consider a theorem due to Noether¹ which is very important in exhibiting the form of an equation in such a way that two pencils can be found such that the product will give the equation.

Theorem. Given two algebraic curves $\phi(x,y)=0$ and $\psi(x,y)=0$. Let one of their interesting points P_i be a multiple point of order q_i for ϕ and of order r_i for ψ ; and suppose further that $q_i < r_i$ and $f(x,y)=0$ a curve which passes through each of the points of intersection of $\phi(x,y)$ and $\psi(x,y)$ and has at P_i a multiple point of order q_i+r_i-1 ; then the equation of $f(x,y)$ can be put in the form

$$f(x,y) = A\phi(x,y) + B\psi(x,y) = 0$$

where $A=0$ and $B=0$ are two other curves of corresponding order. In order for one to give $f(x,y)$ this form it is not necessary that $f(x,y)$ at P_i have a multiple point of order q_i+r_i-1 ; whereas P_i for $f(x,y)$ must be at least of order q_i , the smaller of the two orders; in the later case there must be between the coefficients of the equation $f(x,y)=0$ a linear relation.

Suppose that $\phi(x,y)$ and $\psi(x,y)$ intersect in n points A_i that are regular for each curve, then a curve $f(x,y)=0$ that passes through the A 's as regular points may be written in the form

$$f(x,y) = A\phi(x,y) + B\psi(x,y) = 0$$

where A and B are definite polynomials in x and y . It is now apparent that $f(x,y)=0$ may be generated by two projective pencils

1. Repertorium der Höheren Mathematik II, Vol. (2nd ed.) p.128 Math. Annalen 6, p. 352; ib., 42(1893) 601. ib., 52(1899) 593.

$$A - \lambda B = 0$$

$$\varphi(x, y) + \lambda \phi(x, y) = 0.$$

Any curve that can be written in the form

$$f(x, y) = F(x, y)P(x, y) + G(x, y)Q(x, y)$$

may be generated by two projective pencils

$$F(x, y) - \lambda Q(x, y) = 0,$$

$$G(x, y) + \lambda P(x, y) = 0.$$

$P(x, y)$ and $Q(x, y)$ intersecting in regular points and $f(x, y)$ passing through each of these points.

If we have a given cubic or quartic $f(x, y) = 0$ we can choose four simple distinct points A_1, A_2, A_3, A_4 , on the curve and pass two conics $P(x, y) = 0$ and $Q(x, y) = 0$ through the A 's. Then, according to Noether's theorem it is possible to determine two linear or two quadratic (or one linear and the other quadratic) polynomials $F(x, y) = 0$ and $G(x, y) = 0$ such that

$$F(x, y)P(x, y) + G(x, y)Q(x, y) = 0$$

represents the given cubic, or quartic.

The general proposition can moreover be demonstrated as follows: Choose three distinct points A_1, A_2, A_3 , of an n -ic

$$f(x_1, x_2, x_3) = 0$$

as the coordinate triangle of homogeneous coordinates, so that in the equation the terms x_1^n, x_2^n, x_3^n , are missing. Then it is certainly possible to write f in the form:

$$x_1 \phi(x_1, x_2, x_3) + x_2 \psi(x_1, x_2, x_3) = 0,$$

where ϕ and ψ are ternary forms of order $n-1$. It is now apparent that the given n -ic is generated by the two projective pencils of straight lines and $(n-1)$ -ics

$$x_1 - \lambda x_2 = 0,$$

$$\psi + \lambda \phi = 0.$$

Another method of proving that all cubics and quartics

may be produced by projective pencils is as follows:

It is possible to generate all cubics and quartics by projective pencils of straight lines and conics (including quadratic pencils of lines). Also curves of the n th degree may be generated by projective pencils.

In what follows it will be shown how cubics and quartics are generated.

We shall first consider the cubic.¹

Consider the pencil of conics

$$U + \lambda V = 0,$$

and the pencil of lines

$$\ell + \lambda g = 0$$

These pencils are projective and the locus of their points of intersection is obtained by eliminating λ . The result is

$$Ug - V\ell = 0 \text{ ----- (2)}$$

This equation is of the third degree and is satisfied for $\ell=0$, $g=0$; $u=0$, $v=0$; $u=0$, $\ell=0$; $g=0$, $v=0$. From this it is seen that the product of a pencil of conics and a projective pencil of lines is a curve of the third order which passes through the vertex of the pencil of lines and through the four fundamental points of the pencils of conics.

If we write out the equation (2) in full there will be fourteen constants, but a cubic in its general form in two variables depends upon nine, therefore it is easy to see that it is possible to write any cubic in the form of (2). That is any cubic may be considered as the product of a pencil of conics and a projective pencil of lines.

1. Crelle Journal 71(1870)195.
Emch-Introduction to Proj. Geometry pp. 180.

Quartics.¹ Consider the two pencils

$$\begin{aligned} U + \lambda V &= 0, \text{-----} (3) \\ R + \lambda S &= 0. \end{aligned}$$

U , V , R , and S are of the second degree in x and y and for each value of λ represent two corresponding conics. These conics intersect each other in four points. We obtain the locus of these points by eliminating λ from (3) and get

$$US - VR = 0. \text{-----} (4)$$

This equation is of the fourth degree. Therefore, the product of two projective pencils of conics is a curve of the fourth degree.

If (4) is written out in full, it has twenty constants, but a fourth degree curve depends only upon twelve constants; it is therefore possible as can be shown without difficulty to consider every fourth degree equation as the product of two projective pencils of conics.

As we have seen all cubics and quartics can be produced by projective pencils of straight lines and conics, (including quadratic pencils of lines). We may further state that all n -ics may be generated by projective pencils. This fact follows from Noether's Theorem. For further references see the following:

1. "Kupper, C. Prag. Sb.(1896) (Mth-Nt) No. 1. p.16.
Math. Annal. 48(1897) 401-. ib. 3(1870) 34-.

General Discussion of Quartics and Cubics

Part I

We shall show in what follows as an illustration of the general theory ~~in detail~~ how all cubics and quartics may be generated projectively.

1. The general cubic equation in projective coördinates is

$$x_1 x_2 x_3 + x_1^2 (\ell_1 x_2 + m_1 x_3) + x_2^2 (\ell_2 x_1 + m_2 x_3) + x_3^2 (\ell_3 x_1 + m_3 x_2) = 0 \text{ ----- (1)}$$

If we take three points on this curve as the vertices of the triangle of reference, then the equation (1) must be satisfied by any of the systems:

$$(x_2 = 0, x_3 = 0), (x_3 = 0, x_1 = 0), (x_1 = 0, x_2 = 0)$$

From these we get

$$\ell_1 = 0, \quad m_2 = 0, \quad n_3 = 0,$$

and the equation (1) reduced to

$$x_1 x_2 x_3 + x_1^2 (m_1 x_2 + n_1 x_3) + x_2^2 (n_2 x_3 + \ell_2 x_1) + x_3^2 (\ell_3 x_1 + m_3 x_2) = 0 \text{ ----- (2)}$$

If we put the binomial coefficients of x_1 , x_2 , x_3 , equal to zero we get the equations of the tangents to the cubic at the vertices of the triangle.

If (2) is written in the form

$$x_1 \left[x_2 x_3 + x_1 (m_1 x_2 + n_1 x_3) + \ell_2 x_1^2 + \ell_3 x_3^2 \right] + x_2 x_3 (n_2 x_2 + m_3 x_3) = 0$$

it is at once apparent that this cubic can be generated by the two projective pencils

$$\begin{aligned} x_1 \lambda - (n_2 x_2 + m_3 x_3) &= 0, \\ K + \lambda x_2 x_3 &= 0. \end{aligned} \text{ ----- (3)}$$

K is the quantity in brackets in the above equation.

Thus from (3) we see that any cubic can be generated by a

projective pencil of lines and conics.

2. We shall now form a pair of pencils such that their product will give a cubic having a double point, cusp, or conjugate point at one of the vertices of the triangle of reference.

If in the general equation (1) A, say, is a double point, cusp, or conjugate point, any straight line through A must meet the cubic in two coincident points at A. Any such straight line may be represented by the equation.

$$x_1 = Kx_3.$$

Substitute for x_1 in the general equation, the resulting equation must have two roots equal to zero. Hence the terms involving x_1^3 and x_1^2 must vanish, and we have

$$\ell_1 = 0, \quad m_1 = 0, \quad n_1 = 0.$$

These are the conditions that the locus of the equation (1) should have a double point cusp or conjugate point at A. When these conditions are satisfied we may express (2) in the form

$$x_1(x_2x_3 + \ell_2x_2^2 + \ell_3x_3^2) + x_2x_3(n_2x_2 + m_3x_3) = 0 \quad \text{-----} (4)$$

The two projective pencils then are

$$\begin{aligned} x_1 - \lambda(n_2x_2 + m_3x_3) &= 0 \\ x_2x_3 + \lambda(x_2x_3 + \ell_2x_2^2 + \ell_3x_3^2) &= 0 \end{aligned}$$

Since (4) is the equation of a cubic having either a double point, cusp or conjugate point, we shall show the condition when (4) will be a double point, cusp or conjugate point. If (4) represents a double point the tangents will be real and distinct, if a cusp the tangents will be real and coincident, if a conjugate point the tangents will be imaginary.

Let $x_1 = Kx_3$ be a tangent at A then substituting in (4) the re-

sulting equation is

$$xx_3^2 (fK^2 + gK + h) + x_3^3 (k^3 + mk^2 + nk + r) = 0.$$

This must have all three roots equal ($X_3 = 0$)

Hence $fK^2 + gK + h = 0$ gives two values for K corresponding to the two tangents. The equation of the tangents is

$$fx_2^2 + gx_2x_3 + hx_3 = 0.$$

If the two roots of the quadratic be equal the point A will be a cusp, if they be real and unequal it will be a double point, if imaginary a conjugate point.

3. We shall now show how a circular cubic¹ may be generated by two projective pencils.

The equation for a circular cubic in rectangular coördinates is

$$x^3 + xy^2 + 2fx^2 + 2gxy + hx - 2a(x^2 + y^2) = 0$$

Writing this in the form,

$$(x^2 + y^2)(x - 2a) + x(2fx + 2gy + h) = 0,$$

we see that the two projective pencils

$$\begin{aligned} x^2 + y^2 - \lambda x &= 0, \\ 2fx + 2gy + h + \lambda(x - 2a) &= 0, \end{aligned}$$

are obtained. This shows that the circular cubic can be obtained by taking the product of the above pencils. Thus, we have established the well known fact that every circular cubic may be generated by projective pencils of circles and lines.

4. Every cubic may also be produced by two projective quadratic pencils with a self-corresponding ray.

1. Trans. R. I. A. Vol. XXIV, p. 457.
Casey in this memoir gives an extensive discussion on the circular cubic.

Consider the two quadratic pencils

$$\begin{aligned}ap + \lambda a_1 a_2 &= 0, \\bp + \lambda b_1 b_2 &= 0,\end{aligned}$$

where a , b , a_1 , b_1 , a_2 , b_2 , and p are linear equations. The two pencils have the corresponding ray p in common. If the product of the above pencil is taken we get

$$ab b_1 p - k a_1 a_2 b p = 0,$$

or $p = 0$ and $ab b_2 - k a_1 a_2 b = 0$,

which is a straight line and a cubic.

This cubic passes through the intersection of

$$\begin{aligned}a = 0, b = 0; \quad a = 0, a_2 = 0; \quad a_1 = 0, b_2 = 0; \quad a_1 = 0, b_1 = 0; \quad b = 0, b_2 = 0; \\a_2 = 0, b_1 = 0; \quad a_2 = 0, b_2 = 0; \quad b = 0, b_1 = 0; \quad b_1 = 0, b_2 = 0;\end{aligned}$$

In all, nine points, which determine a cubic. The coefficients in a, a_1, a_2, b, b_1, b_2 may always be chosen such that the above equation represents that of any cubic.

5. By Noether's Theorem, which we have given in the introduction, it follows that a quartic can be produced by two projective pencils of conics. We shall now show the condition and form of the quartic, and also the pencils of conics which will give a quartic with three double points, or deficiency zero.

Allowing the quartic to pass through the vertices of the triangle of reference the terms involving the fourth powers of x_1, x_2, x_3 , will be absent and the equation for the quartic will be

$$f(x_1, x_2, x_3) = a_{3,1,2} x_1^3 x_2 + a_{3,1,3} x_1^3 x_3 + a_{3,2,1} x_2^3 x_1 + a_{3,2,3} x_2^3 x_3 + a_{3,3,1} x_3^3 x_1 + a_{3,3,2} x_3^3 x_2 + a_{2,1,2} x_1^2 x_2^2 + a_{2,1,3} x_1^2 x_2 x_3 + a_{2,2,1} x_1 x_2^2 x_3 + a_{2,2,3} x_1 x_2 x_3^2 + a_{2,3,1} x_1 x_3^2 x_2 + a_{2,3,2} x_1 x_3 x_2^2 + a_{1,2,3} x_1 x_2 x_3^2 + a_{1,3,2} x_1 x_3 x_2^2 + a_{1,2,1} x_1^2 x_2 x_3 + a_{1,3,1} x_1^2 x_3 x_2 + a_{1,2,2} x_1 x_2^2 x_3 + a_{1,3,3} x_1 x_3^2 x_2 = 0$$

Let $A_1 = (x_1, 0, 0)$, $B = (0, x_2, 0)$, $C = (0, 0, x_3)$ be the vertices of the triangle of reference. In order for these points to be double points they must be solutions of the three following equations

$$\frac{\partial f}{\partial x_1} = a_{\frac{3}{3}\frac{2}{2}\frac{3}{3}}x_2^3 + a_{\frac{4}{4}\frac{3}{3}\frac{3}{3}}x_2^3 + 3a_{\frac{5}{5}\frac{2}{2}\frac{2}{2}}x_2^2 + 3a_{\frac{6}{6}\frac{2}{2}\frac{2}{2}}x_2^2 + 2a_{\frac{7}{7}\frac{2}{2}\frac{2}{2}}x_2^2 + 2a_{\frac{8}{8}\frac{2}{2}\frac{2}{2}}x_2^2 + 2a_{\frac{9}{9}\frac{2}{2}\frac{2}{2}}x_2^2 + a_{\frac{10}{10}\frac{2}{2}\frac{2}{2}}x_2^2 + a_{\frac{11}{11}\frac{2}{2}\frac{2}{2}}x_2^2 = 0, \text{-----} (2)$$

$$\frac{\partial f}{\partial x_2} = 3a_{\frac{3}{3}\frac{2}{2}\frac{3}{3}}x_2^2 + a_{\frac{4}{4}\frac{3}{3}\frac{3}{3}}x_2^2 + a_{\frac{5}{5}\frac{3}{3}\frac{3}{3}}x_2^2 + 3a_{\frac{6}{6}\frac{2}{2}\frac{2}{2}}x_2^2 + 2a_{\frac{7}{7}\frac{2}{2}\frac{2}{2}}x_2^2 + 2a_{\frac{8}{8}\frac{2}{2}\frac{2}{2}}x_2^2 + a_{\frac{9}{9}\frac{2}{2}\frac{2}{2}}x_2^2 + 2a_{\frac{10}{10}\frac{2}{2}\frac{2}{2}}x_2^2 + a_{\frac{11}{11}\frac{2}{2}\frac{2}{2}}x_2^2 = 0, \text{-----} (3)$$

$$\frac{\partial f}{\partial x_3} = 3a_{\frac{3}{3}\frac{2}{2}\frac{3}{3}}x_2^2 + 3a_{\frac{4}{4}\frac{2}{2}\frac{3}{3}}x_2^2 + a_{\frac{5}{5}\frac{2}{2}\frac{3}{3}}x_2^2 + a_{\frac{6}{6}\frac{2}{2}\frac{3}{3}}x_2^2 + 2a_{\frac{7}{7}\frac{2}{2}\frac{3}{3}}x_2^2 + 2a_{\frac{8}{8}\frac{2}{2}\frac{3}{3}}x_2^2 + a_{\frac{9}{9}\frac{2}{2}\frac{3}{3}}x_2^2 + a_{\frac{10}{10}\frac{2}{2}\frac{3}{3}}x_2^2 + 2a_{\frac{11}{11}\frac{2}{2}\frac{3}{3}}x_2^2 = 0. \text{-----} (4)$$

By substituting the coordinates of the points A, B, and C in (2),

(3), (4) we can find the relation between the a's. For the point A

$$\frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} = a_{\frac{3}{3}\frac{2}{2}\frac{3}{3}}x_2^3 = 0, \quad \frac{\partial f}{\partial x_3} = a_{\frac{7}{7}\frac{2}{2}\frac{3}{3}}x_2^3 = 0.$$

For the point B

$$\frac{\partial f}{\partial x_1} = a_{\frac{3}{3}\frac{2}{2}\frac{3}{3}}x_2^3 = 0, \quad \frac{\partial f}{\partial x_2} = 0, \quad \frac{\partial f}{\partial x_3} = a_{\frac{8}{8}\frac{2}{2}\frac{3}{3}}x_2^3 = 0.$$

For the point C

$$\frac{\partial f}{\partial x_1} = a_{\frac{4}{4}\frac{2}{2}\frac{3}{3}}x_2^3 = 0, \quad \frac{\partial f}{\partial x_2} = a_{\frac{6}{6}\frac{2}{2}\frac{3}{3}}x_2^3 = 0, \quad \frac{\partial f}{\partial x_3} = 0.$$

From these equations we see that

$a_3, a_4, a_6, a_7, a_8, a_9$, must be zero.

Hence (1) takes the form

$$b_{\frac{0}{0}\frac{2}{2}\frac{2}{2}}x_1^2 + b_{\frac{1}{1}\frac{2}{2}\frac{2}{2}}x_1^2 + b_{\frac{2}{2}\frac{2}{2}\frac{2}{2}}x_1^2 + b_{\frac{3}{3}\frac{2}{2}\frac{2}{2}}x_1^2 + b_{\frac{4}{4}\frac{2}{2}\frac{2}{2}}x_1^2 + b_{\frac{5}{5}\frac{2}{2}\frac{2}{2}}x_1^2 = 0. \text{-----} (5)$$

To show how (5) can be produced by two projective pencils, write it in the form

$$x_1^2 (b_{\frac{0}{0}\frac{2}{2}\frac{2}{2}} + b_{\frac{1}{1}\frac{2}{2}\frac{2}{2}}x_3 + b_{\frac{2}{2}\frac{2}{2}\frac{2}{2}}x_3) + x_2^2 (b_{\frac{3}{3}\frac{2}{2}\frac{2}{2}} + b_{\frac{4}{4}\frac{2}{2}\frac{2}{2}}x_3 + b_{\frac{5}{5}\frac{2}{2}\frac{2}{2}}x_3) = 0$$

from which it follows that two pencils are

$$\begin{aligned} x_1^2 - \lambda N &= 0 \\ x_2^2 + \lambda M &= 0 \end{aligned}$$

where M is the coefficient of x_1^2 and N is coefficient of x_2^2 .

6. If we wish to form a quartic having two double points, or deficiency one, say at A and B of the triangle of reference we must have $A = (x_1, 0, 0)$, $B = (0, x_2, 0)$ as solutions of the partial differential equations (2), (3), (4).

From these we find that

$$a_3, a_5, a_7, a_8,$$

must be zero. Hence the general equation of the quartic takes the form

$$axx^3_{413} + axx^3_{623} + axx^2_{912} + axx^2_{1013} + axx^2_{1123} + axx^2_{12123} + axxx_{13213} + axxx_{14312} = 0, \text{-----} (6)$$

To show how (6) can be produced by two projective pencils of conics, write it in the form

$$x_3^2 \left[axx_{413} + axx_{623} + ax^2_{101} + ax^2_{112} \right] + xx_{12} \left[axx_{912} + axx_{1213} + axx_{1323} + ax^2_{143} \right] = 0.$$

From this it follows that

$$\begin{aligned} M - \lambda xx_{12} &= 0, \\ N + \lambda x^2_{143} &= 0, \end{aligned}$$

are the required pencils. M and N are the coefficients of x_3^2 and xx_{12} respectively.

Consider now the quartic having one double point, say at $A = (x, 0, 0)$. This point must satisfy the equations (2), (3), (4). From these we find that a_5 and a_7 are zero. The form of the quartic will be

$$axx^3_{312} + axx^3_{413} + axx^3_{623} + axx^2_{912} + axx^2_{1013} + axx^2_{1123} + axxx_{12123} + axxx_{13213} + axxx_{14312} = 0 \text{-----} (7)$$

To show how (7) can be produced by two projective pencils of conics, consider the two pencils

$$\begin{aligned} \phi_1 + \lambda \phi_2 &= 0 \text{-----} (8) \\ \psi_1 + \lambda \psi_2 &= 0 \end{aligned}$$

in which

$$\begin{aligned} \phi_1 &= bxx_{312} + bxx_{123} + axxx_{231} = 0 \\ \psi_1 &= cxx_{312} + cxx_{231} + cxx_{123} = 0 \\ \phi_2 &= dx^2_{12} + dx^2_{33} + \cancel{f}xx_{12} + \cancel{f}xx_{23} + \cancel{f}xx_{31} \\ \psi_2 &= \ell x^2_{12} + \ell x^2_{33} + \delta xx_{12} + \delta xx_{23} + \delta xx_{31} \end{aligned}$$

If we take the product of (8) we get

$$\phi_1 \psi_2 - \phi_2 \psi_1 = 0 \text{-----} (9)$$

This equation corresponds to (7) and the a's in (7) can be

found in terms of the constants in (9). Hence it follows that a quartic having one double point can be produced by two projective pencils of conics.

In the constructive theory of binodal quartics it is also important to know that every binodal quartic may be generated by two projective quadratic pencils with their vertices at the double-points of the quartic.

Special Cubics and Quartics

Part II

1. Consider the pencil of circles

$$(x-\lambda)^2 + y^2 = \lambda^2 \text{-----(1)}$$

and the pencil of lines passing through the point $(0,1)$

with slope $\frac{1}{\lambda}$. The equation of the pencil of lines is

$$-\lambda + x + \lambda y = 0. \text{-----(2)}$$

If λ is eliminated between (1) and (2) the following cubic is derived

$$f(xy) = y^3 + xy^2 - y^2 + x^2 = 0. \text{-----(3)}$$

In this case the two base points of the pencil of circles are coincident at the origin. Therefore, the origin is a double point of the cubic. The slopes of the tangents at the origin are ± 1 .

This cubic is circular, for its equation can be written in the form

$$(x^2 + y^2)y + x^2 - y^2 = 0$$

It follows that it is a circular cubic, from the fact that it passes through the base points of the pencil of circles which also pass through the circular points. The cubic also passes through these, and is therefore circular.

According to a theorem in Part I a circular cubic may be generated in an infinite number of ways by projective pencils of circles and pencils of lines. Take for example the double point of the cubic and any other point of the cubic as base-points of the pencil of circles. For this purpose we can put the cubic in the form

$$(x-1)^2 + y^2 = 1^2$$

$$-1 + x + 1y = 0$$

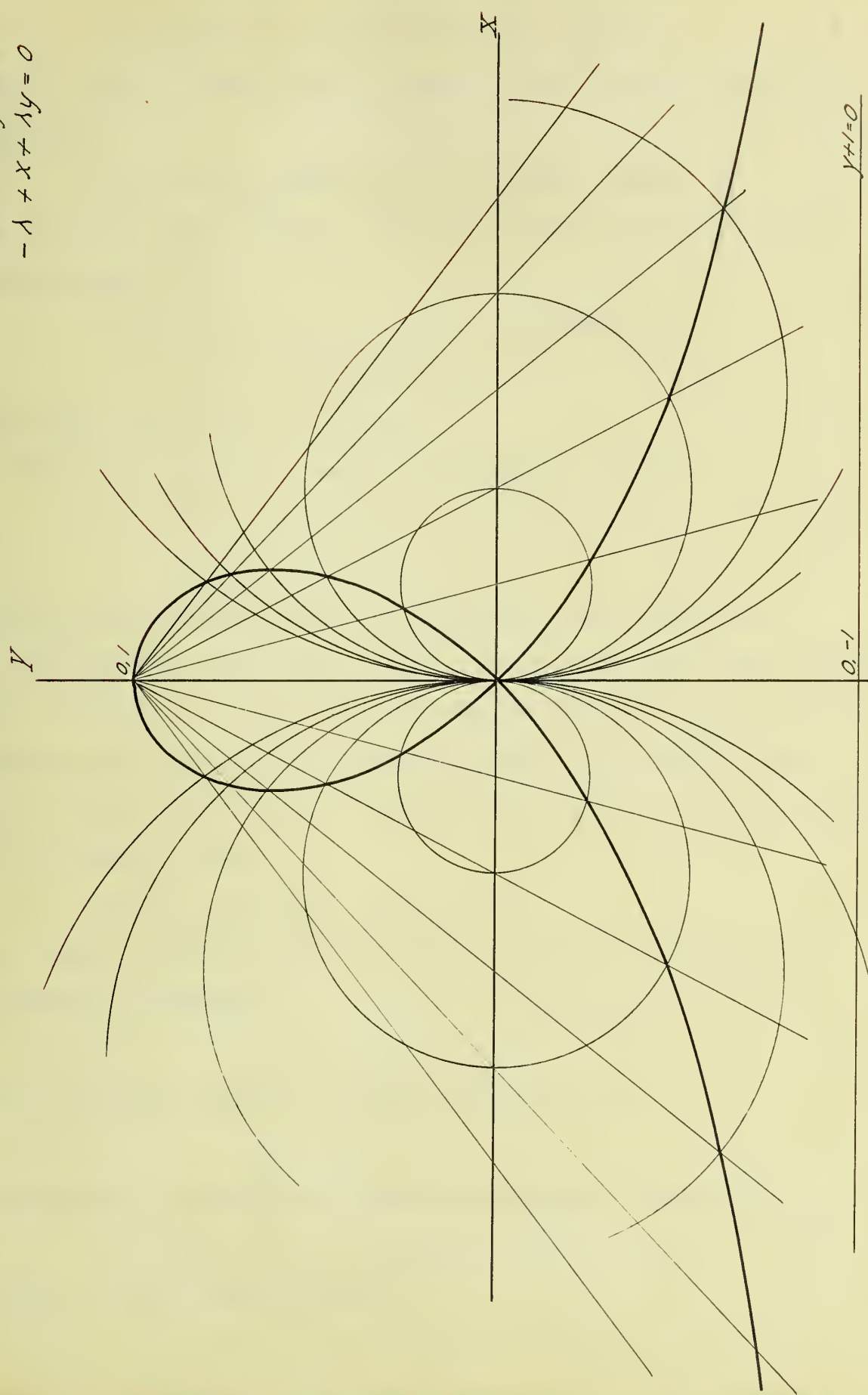


Fig. I

$$(x^2 + y^2 - 2\alpha x - 2\beta y)y - y^2 + x^2 + 2\alpha xy + 2\beta y^2 = 0.$$

Now $x^2 + 2\alpha xy + y^2(2\beta - 1)$ may be resolved into the two linear factors

$$\left[x + (\alpha + \sqrt{\alpha^2 - 2\beta - 1})y \right] \left[x + (\alpha - \sqrt{\alpha^2 - 2\beta - 1})y \right],$$

so that the cubic appears now as the product of the two projective pencils

$$x^2 + y^2 - 2\alpha x - 2\beta y + \lambda \left[x + (\alpha + \sqrt{\alpha^2 - 2\beta - 1})y \right] = 0,$$

$$x + (\alpha - \sqrt{\alpha^2 - 2\beta - 1})y - \lambda y = 0,$$

in which α and β are arbitrary.

When $\alpha = 0$, $\beta = 0$, we have the projective pencils

$$x^2 + y^2 + \lambda(x + y) = 0$$

$$x - y - \lambda y = 0$$

which permit another simple construction for the cubic.

To construct the locus of (3), lay off on the x axis values of λ . With $(\lambda, 0)$ as a center with radius λ construct circles. Through the point $(0, 1)$ and the point $(\lambda, 0)$ draw lines. Where these lines cut the corresponding circles will be points on the graph of (3).

Figure I shows the locus of (3).

2. Another cubic may be generated by taking the pencil of concentric circles

$$x^2 + y^2 = \lambda$$

and the pencil of lines through the point $(0, 1)$

$$\frac{x}{\lambda} + y = 1.$$

Eliminating λ between the two pencils there is obtained

$$(x^2 + y^2)(1 - y) - x = 0, \text{ -----(4)}$$

By aid of the transformation

$$x = \frac{x'}{\lambda' - 1}$$

$$y = \frac{y'}{\lambda' - 1}$$

we get a cubic

$$(x^2 + y^2)(x - y - 1) - x(x - 1) = 0$$

which clearly passes through the points

$$I_1 \begin{cases} x = 1 \\ y = i \end{cases} \quad I_2 \begin{cases} x = 1 \\ y = -i \end{cases}$$

Therefore the original cubic passes through the circular points which is also apparent from equation (4).

The points of inflexion of (4) are collinear. This can be shown by solving the cubic and its second derivative, equated to zero, as simultaneous equations. This cubic has $y - 1 = 0$ as an asymptote. It has no double point, since there is no solution to

$$\frac{\partial f}{\partial x} = 0, \text{ and } \frac{\partial f}{\partial y} = 0.$$

It is therefore of deficiency one. The derivative of (4) is

$$\frac{dy}{dx} = \frac{1 + axy - 2x}{2y - x^2 - 3y^2}$$

From this equation it is seen that the slopes of the tangents at the origin and at the points

$$(1, 0), (0, 1)$$

are respectively ∞ , 1, and -1.

To construct the locus of (4) it is convenient to take for values of λ numbers which are perfect squares. Let λ equal 4, say. Then the radius of the circle is 2. Joining the points (0, 1) and (4, 0). This line cuts the circle at P and P' which

$$x^2 + y^2 = 1$$

$$\frac{x}{x} = 1$$

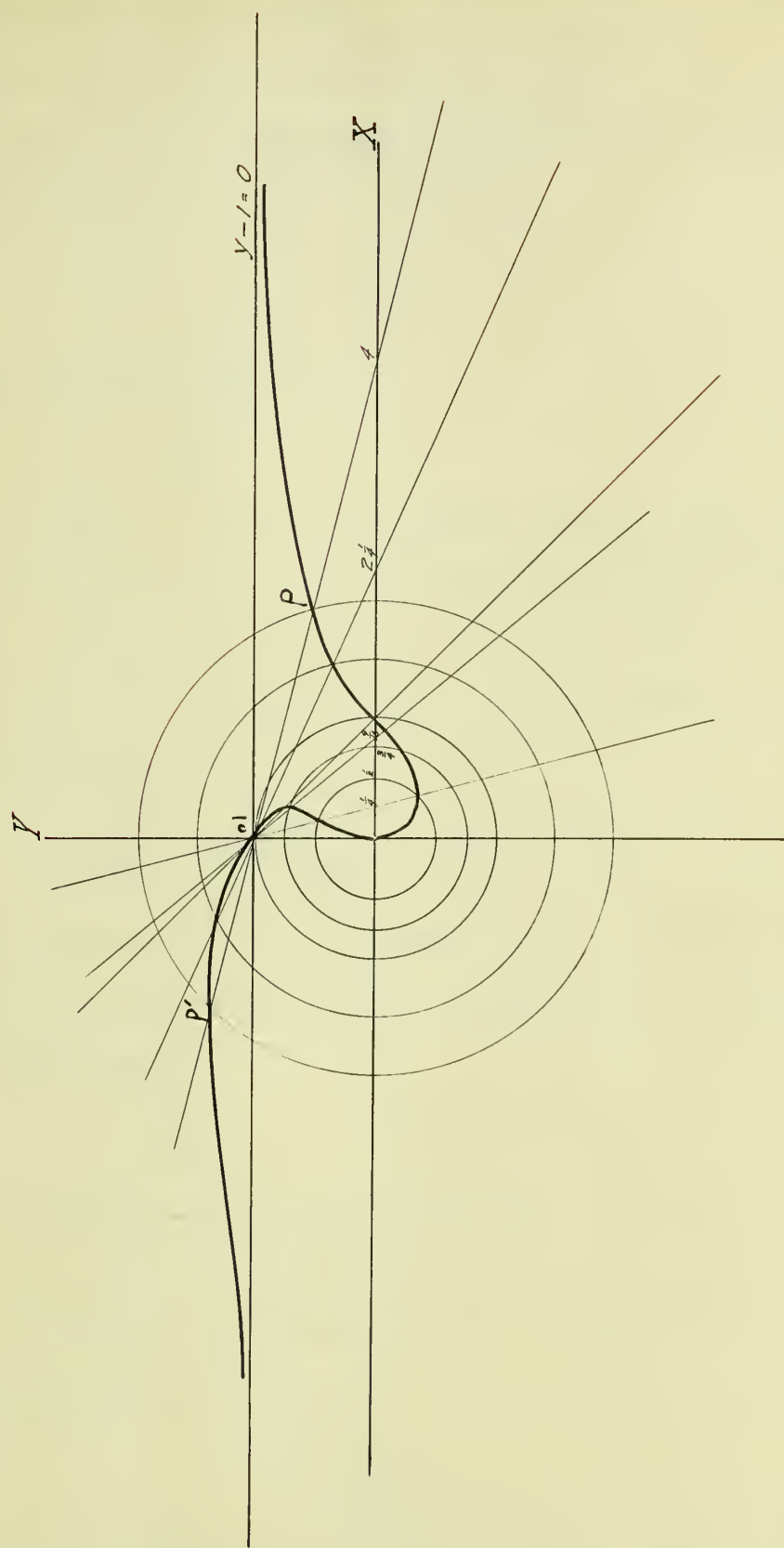
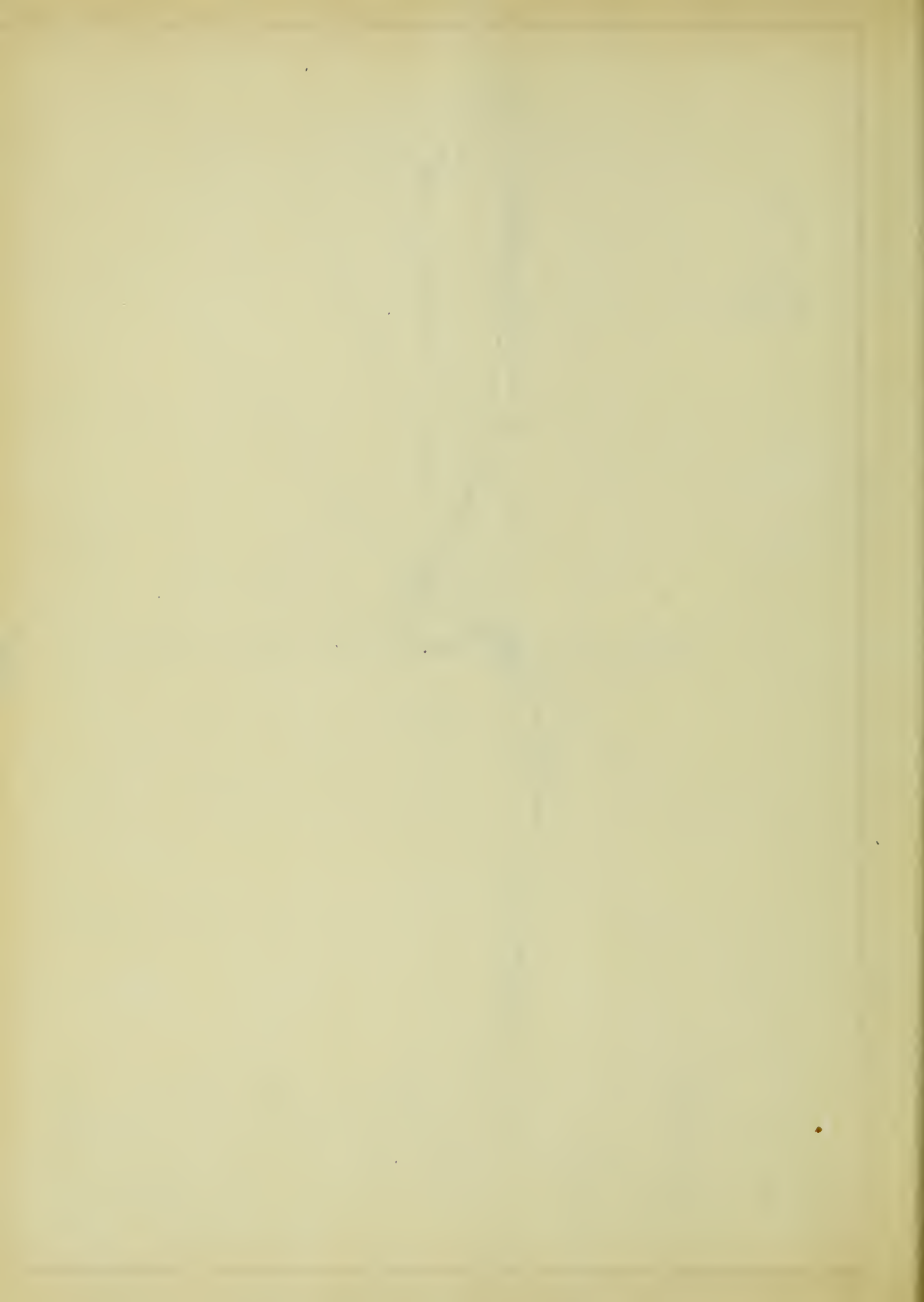


Fig. II



are points on the locus of (4).

Figure II exhibits the locus of (4).

3. Another cubic may be generated by taking the product of a pencil of parabolas and a pencil of lines.

Let

$$y^2 = \lambda x$$

be the pencil of parabolas, and

$$\frac{x}{\lambda} + y = 1$$

the pencil of lines. If λ is eliminated between these two equations there is obtained the following 3rd degree equation

$$y^3 - y^2 + x^2 = 0. \text{ ----- (5)}$$

This curve has a double point at the origin, since (0,0) is a simultaneous solution of

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0, \text{ and of (5).}$$

The curve (5) has a point of inflexion at the infinite point of the X axis. This can be shown by making use of the perspective transformation

$$x = \frac{x'}{x'-1}, \quad y = \frac{y'}{x'-1},$$

from which we get as a resulting equation

$$y^3 - y^2(x-1) + x^2(x-1) = 0$$

If we find the second derivative of this equation and put it equal to zero, the point

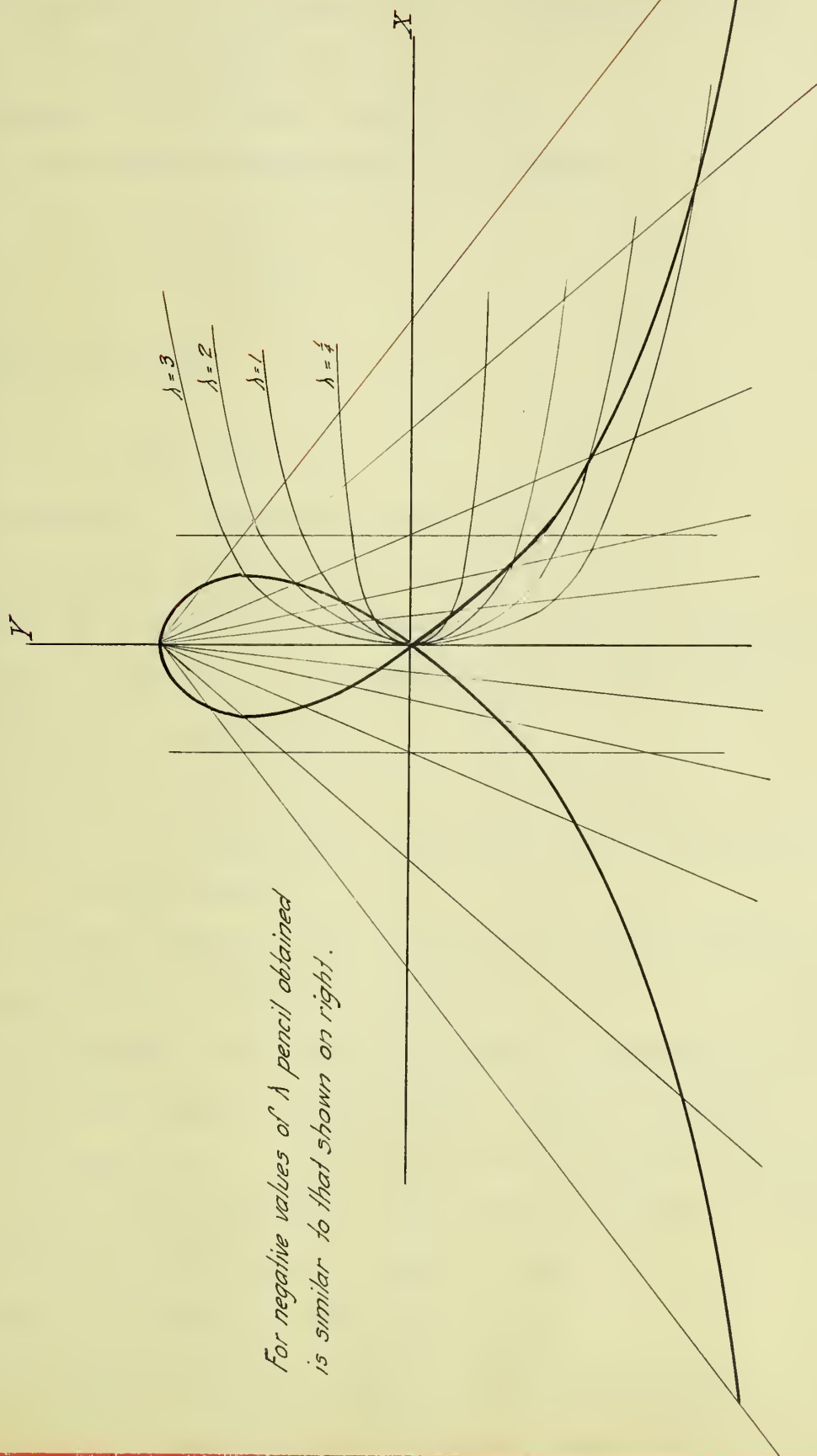
$$I. \quad \begin{cases} x = 1 \\ y = 0 \end{cases}$$

will satisfy the differential equation. Hence the point at infinity is a point of inflexion.

This curve is constructed by laying off different values

$$y^2 = \lambda x$$

$$\frac{x}{\lambda} + y = 1$$



For negative values of λ pencil obtained is similar to that shown on right.

Fig. III

of λ on the X axis. Through the points $(0,1)$ and $(\lambda,0)$ draw the pencil of lines. For each value of λ given construct the parabola. Then note the points of intersection of corresponding pairs. These points will be on the locus of (5).

Figure III shows the locus of (5).

4. Consider now the following pencil of cissoids

$$y^2 = \frac{x^3}{2\lambda - x},$$

and the pencil of lines

$$\frac{x}{\lambda} + y = 1.$$

Taking the product of these two pencils we have

$$y^2 x + 2xy^2 + xy^3 - xy^2 - x^3 = 0, \text{ ----- (6)}$$

This equation is of the fourth degree, but it can be factored into a linear and cubic factor. The linear factor is the y-axis. Writing the other factor in the form

$$ox^3 + x^2(y-1) + y^3 + y^2 = 0 \text{ ----- (6)}$$

we see $y-1=0$ is an asymptote. Putting $y=mx$ in (10) we have

$$x^2 (mx^3 + mx + m^2 - 1) = 0$$

This shows that two branches pass through the origin and the slopes of the tangents at the origin are 1 and -1.

To construct the locus of the above equation lay off values of λ on the x axis. Construct the cissoid corresponding to each value ~~of~~ given to λ . Also draw lines through the points $(0,1)$ and $(\lambda,0)$, the points where each line intersects its corresponding cissoid will be points of the locus of (6).

Figure IV exhibits the locus of (6).

5. It is sometimes convenient, when a certain equation is given,

$$y^2 = \frac{x^3}{2x - x}$$

$$\frac{x}{y} + y = 1$$

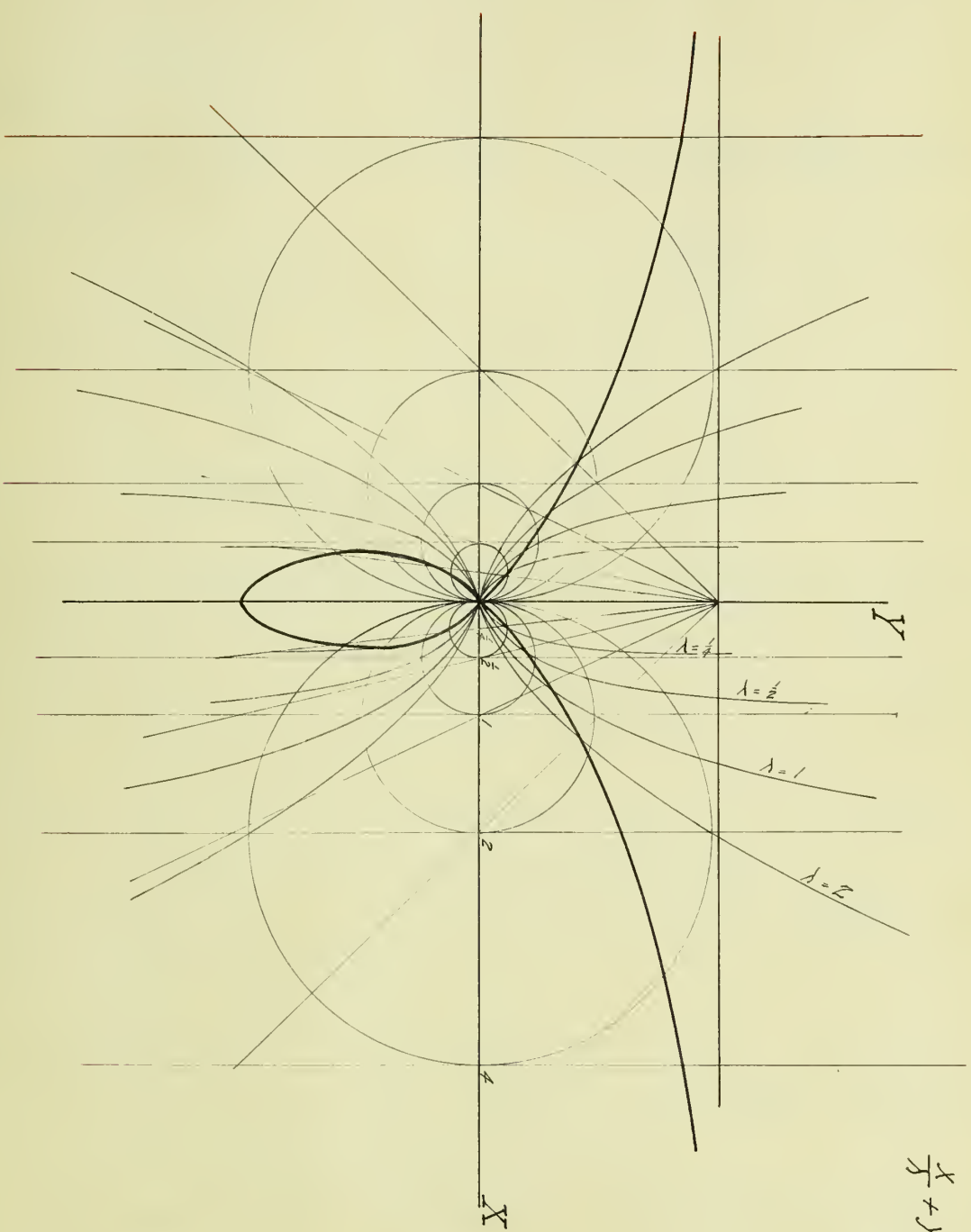


Fig. III

to find two pencils whose product will give the equation.

Suppose the equation

$$x^4 - 4xy^2 + y^3 = 0 \text{ ----- (7)}$$

is given, and it is desired to find two pencils whose product will give the equation (7).

Writing (7) in the form

$x^2(x^2 - 2y) - y(2x^2 - y^2) = 0$, it is at once seen that the following two pencils

$$\begin{aligned} x^2 - 2y - \lambda y &= 0 \\ (2 - \lambda)x^2 - y^2 &= 0 \end{aligned} \text{ ----- (7a)}$$

will give (7). Writing (7a) in the form

$$y(2 + \lambda) = x^2 \quad \text{and}$$

$$\left(\frac{1}{\sqrt{2 - \lambda}}x + y \right) \left(\frac{1}{\sqrt{2 - \lambda}}x - y \right) = 0$$

we have a pencil of parabolas, and a pencil of degenerate hyperbolas. From these equations is seen, in order to have real curves, that λ is restricted to the following condition $\lambda \leq 2$.

If $y = mx$ is substituted in (7) we have $x^3(x - 4m + m^3) = 0$. --- (8)
This equation has three of its roots each equal to zero. Hence there are three branches passing through the origin, that is the origin is a triple point. Consider next the second factor of (8)

$$x - 4m + m^3 = 0.$$

Putting $x = 0$ we have for values of m ; 0, 2 and -2. These values of m are the slopes of the three tangents to the three branches passing through the origin.

To construct the locus of (7) construct the two pencils (7a). The points of intersection of corresponding elements of the two pencils will be points on the locus of (7). For example, let $\lambda = 1$, then the first of (7a) is

$$y(2+\lambda) = x^2$$

$$(\sqrt{2-\lambda} \cdot x + y)(\sqrt{2-\lambda} \cdot x - y) = 0$$

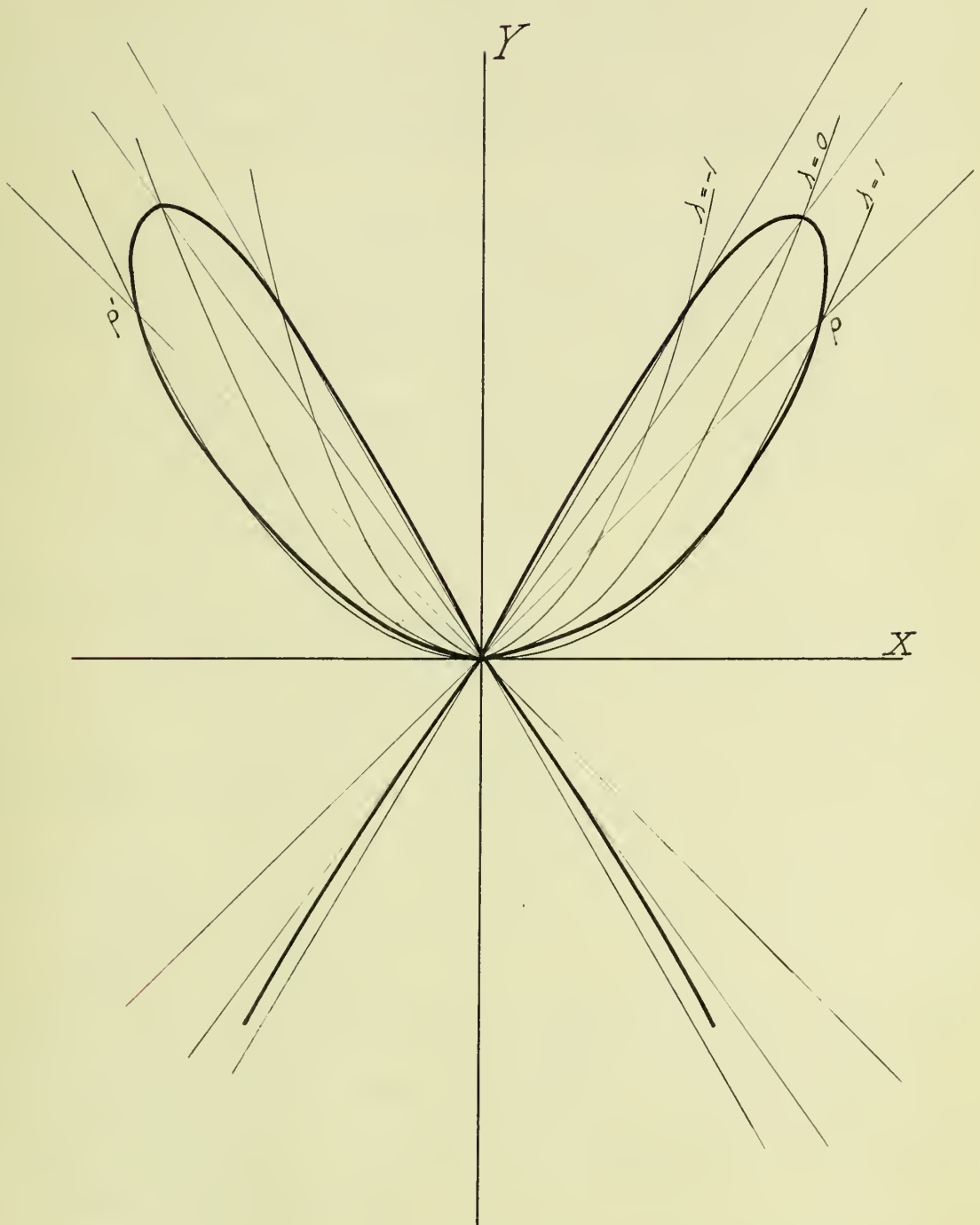


Fig. V

$$3y = x^2$$

which is a parabola. The second of (7a) becomes $(x+y)(x-y) = 0$, which are two lines. Where the parabola intersects these two lines, as at P and P', will be points on the locus of (7). In a similar way other points can be constructed by giving various values to λ .

Figure V exhibits the locus of (7).

6. We shall now take the equation of the lemniscate

$$(x^2 + y^2)^2 = x^2 - y^2 \text{-----(9),}$$

and show how to form two projective pencils such that their product will give the lemniscate. First writing (9) in the form

$$x^4 + 2x^2y^2 + y^4 - x^2 + y^2 = 0,$$

and dividing by x^2y^2 we get

$$\frac{2y^2 + x^2 - 1}{y^2} + \frac{y^2 + 1}{x^2} = 0.$$

Putting $\lambda = \frac{y^2 + 1}{x^2} = \frac{1 - x^2 - 2y^2}{y^2}$ we get

after simplifying

$$\begin{array}{rcl} x^2 + y^2(2 - \lambda) & = & 1 \\ \lambda x^2 + y^2 & = & -1 \end{array}$$

for the two pencils such that their product will give (9). The first pencil will give a series of ellipses for values of $\lambda < 2$. For values greater than 2 a series of hyperbolas. In the second we get a pencil of hyperbolas.

The lemniscate could be constructed by constructing the two pencils already found, and then noting the points of intersection of corresponding elements. But we shall take two other pencils which will give, by taking their products, the equation for the

$$\left(x - \frac{\lambda}{2}\right)^2 + \left(y + \frac{\lambda}{2}\right)^2 = \frac{\lambda^2}{2}$$

$$\left(x - \frac{\lambda}{2\lambda}\right)^2 + \left(y - \frac{\lambda}{2\lambda}\right)^2 = \frac{\lambda^2}{2\lambda^2}$$

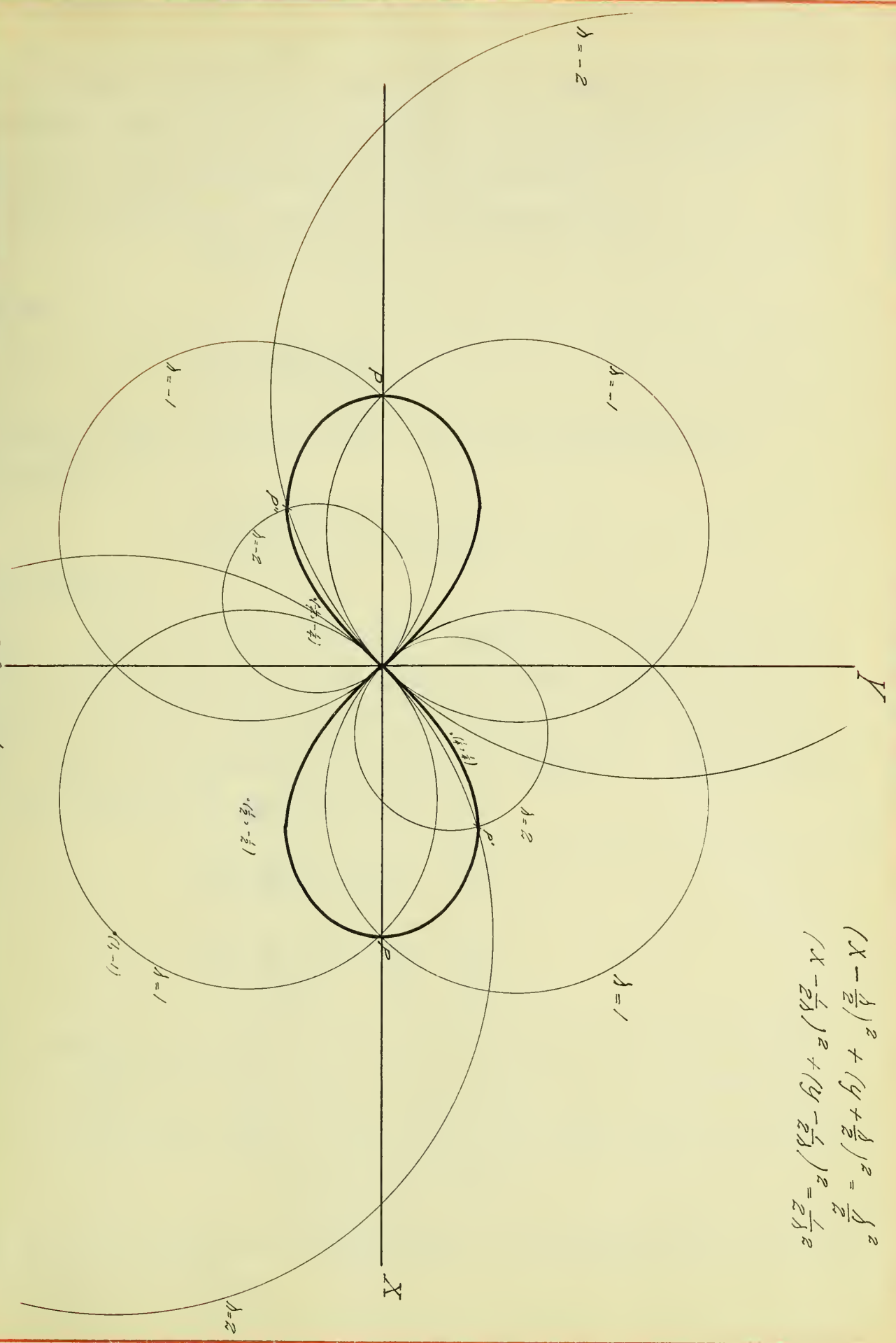


Fig. V'

lemniscate. We might say here, that if we have given a certain equation, there are in most cases various projective pencils such that their product will give the equation.

Let us divide the equation of the lemniscate by

$$(x^2+y^2)(x-y).$$

We get

$$\frac{x^2 + y^2 x + y}{x - y x^2 + y^2}.$$

If we put these equations equal to λ , the following two equations,

$$\begin{aligned} x^2 + y^2 - \lambda(x-y) &= 0, \\ \lambda(x^2 + y^2) - x - y &= 0, \end{aligned}$$

are obtained.

These are two projective pencils of circles such that their product will give the lemniscate. These pencils are more easily constructed than those first derived. Their construction and that of the lemniscate are shown in Figure V. We see that if the pencils are written in the form,

$$\left(x - \frac{\lambda}{2}\right)^2 + \left(y + \frac{\lambda}{2}\right)^2 = \frac{\lambda^2}{2}$$

$$\left(x - \frac{1}{2\lambda}\right)^2 + \left(y - \frac{1}{2\lambda}\right)^2 = \frac{1}{2\lambda^2}$$

the centers of the first lie on a line bisecting the second and fourth quadrants, while those of the second lie on a line bisecting the first and third. This can also be seen from the relation

$$\frac{\lambda}{2} \cdot \frac{1}{2\lambda} = \text{const.}$$

by which the distances of the centers of corresponding circles are related by inversion.

7. Another curve that is frequently met with in the calculus is the one generated by the following pencils:

$$\frac{y}{x} \frac{(1-\lambda)}{(\lambda-1)} + x = 0 \text{-----}(10)$$

$$-y = 0.$$

If λ is eliminated between (10) we have after simplifying

$$y^2 - x^3 = 0,$$

which is called the semi-cubical parabola. From (10) it is seen that for all values of $\lambda < 1$ the points on the lower branch are obtained, while for all values of $\lambda > 1$ the points on the upper branch result.

To obtain (10) from

$$y^2 - x^3 = 0 \text{-----}(11)$$

write (11) in the form

$$y(y+x) - x(y+x^2) = 0.$$

From this equation it is seen that the two pencils

$$\frac{y+x}{x} = \lambda \text{ and } \frac{y+x^2}{y} = \lambda$$

when simplified give (11).

In equations (10) a geometrical significance can easily be given, If we put

$$\lambda - 1 = u$$

the first pencil becomes

$$y = ux,$$

and the second

$$x^2 = uy.$$

The first is a pencil of lines passing through the origin having the slope u , and the second a pencil of parabolas passing through the origin with u as parameter.

The same cubic may also be generated by the two projective pencils

$$y + x - \lambda x^2 = 0,$$

$$x - 1 + \lambda(y-x) = 0;$$

i. e. by another pencil of paraboles and a projective pencil of lines with the vertex $(1, 1)$.

8. For another example we shall consider the two pencils

$$x^2 + 2x + y^2(2 - \lambda) = 0, \text{-----} (13)$$

$$\lambda x + y^2 + 2x - 1 = 0.$$

Writing these in forms

$$\begin{aligned} (x + 1) + \frac{y^2}{2 - \lambda} &= 1, \\ \frac{(x + \frac{1}{\lambda})^2}{\frac{\lambda + 1}{\lambda^2}} + \frac{y^2}{1 + \frac{1}{\lambda}} &= 1, \end{aligned} \quad \left. \begin{array}{l}) \\) \\) \\) \\) \\) \\) \\) \\) \end{array} \right\} \text{-----} (14)$$

it is seen that the first is an ellipse with center at $(-1, 0)$ and axes 1 and $\frac{1}{\sqrt{2 - \lambda}}$ for all values of λ satisfying the

inequality. $-\infty < \lambda < 2$.

If $\lambda > 2$ a

system of hyperboles is obtained.

In the second pencil

for all positive values of λ and all negative values greater than -1 a system of ellipses is obtained with center

$$\left(-\frac{1}{\lambda}, 0\right) \text{ and semi-axes } \frac{1}{\lambda} \sqrt{\lambda+1}, \sqrt{1+\frac{1}{\lambda}}.$$

If λ is eliminated between (13) we have after simplifying

$$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0. \text{-----(15)}$$

In (15) replace x and y respectively by

$$\begin{aligned} x &= r \cos \theta, \\ y &= r \sin \theta. \end{aligned}$$

On simplifying we have

$$r = 1 - \cos \theta.$$

This equation is that of the Cardioid.

We shall show now how (15) can be produced by two projective pencils of circles. Writing (15) in the form

$$(x^2 + y^2)^2 + (x^2 + y^2)(2x - 1) + x^2 = 0,$$

$$\text{or } (x^2 + y^2)(x^2 + y^2 + 2x - 1) + x^2 = 0, \text{-----(15a)}$$

it is at once seen that

$$\begin{aligned} x^2 + y^2 - \lambda x &= 0, \\ x + \lambda(x^2 + y^2 + 2x - 1) &= 0, \end{aligned}$$

are the required pencils.

To find out in what manner (15a) passes through the circular points, we shall transform it by means of

$$x = \frac{x'}{x'-1}, \quad y = \frac{y'}{x'-1}.$$

The resulting equation after simplifying and dropping primes is

$$(x^2 + y^2) \left[x^2 + y^2 + 2x(x-1) - (x-1)^2 \right] + x^2(x-1)^2 = 0. \text{-----(15b)}$$

Now pass a line $y - i = m(x - 1)$ through the point $(1, i)$, so that

$$x^2 + y^2 = (x-1) \left[x + 1 + 2mi + m^2(x-1) \right],$$

and after substituting in (15b), $(x-1)^2$ appears as a factor.

This shows that (1,1) is a multiple point of order 2. To determine its nature more fully (15b), after division by $(x-1)^2$ becomes

$$[x+1+2mi+m^2(x-1)] [x+1+2mi+m^2(x-1)+2x-x+1]+x^2=0.$$

When $x=1$, the secant approaches the tangent and we have for its slope m

$$\text{or } \begin{aligned} 4(1+mi)^2 + 4(1+mi) + 1 &= 0, \\ (2mi+3)^2 &= 0. \end{aligned}$$

The slopes at the multiple point are therefore

$$m_1 = m_2 = \frac{3i}{2}.$$

It is therefore a cusp. Hence the cardioid has the circular points as cusps.

9. We shall next consider the two pencils

$$\frac{x^2 + 2y^2 - 4x + 3}{y^2} = \lambda,$$

$$\text{and } \frac{4x - y^2 + 1}{x^2} = \lambda.$$

Writing these in the form

$$\begin{aligned} (x-2)^2 + y^2(2-\lambda) &= 1, \\ \left(x - \frac{2}{\lambda}\right)^2 + \frac{y^2}{\frac{4+\lambda}{\lambda^2}} &= 1, \end{aligned} \quad \begin{aligned} & \\ & \\ & \text{-----} (16) \\ & \\ & \end{aligned}$$

the first is seen to be an ellipse for all values of $\lambda < 2$, for values of $\lambda > 2$, a pencil of hyperbolas is formed. The center in both cases being (2,0) and axes 1 and $\frac{1}{\sqrt{2-\lambda}}$. In the second a

pencil of ellipses is formed for all positive values of λ . For negative values a pencil of hyperbolas is obtained. For large

$$(X-2)^2 + Y^2 / (2-\lambda) = 1$$

$$\frac{(X-\frac{2}{\lambda})^2}{\frac{4+\lambda}{\lambda^2}} + \frac{Y^2}{\frac{4+\lambda}{\lambda}} = 1$$

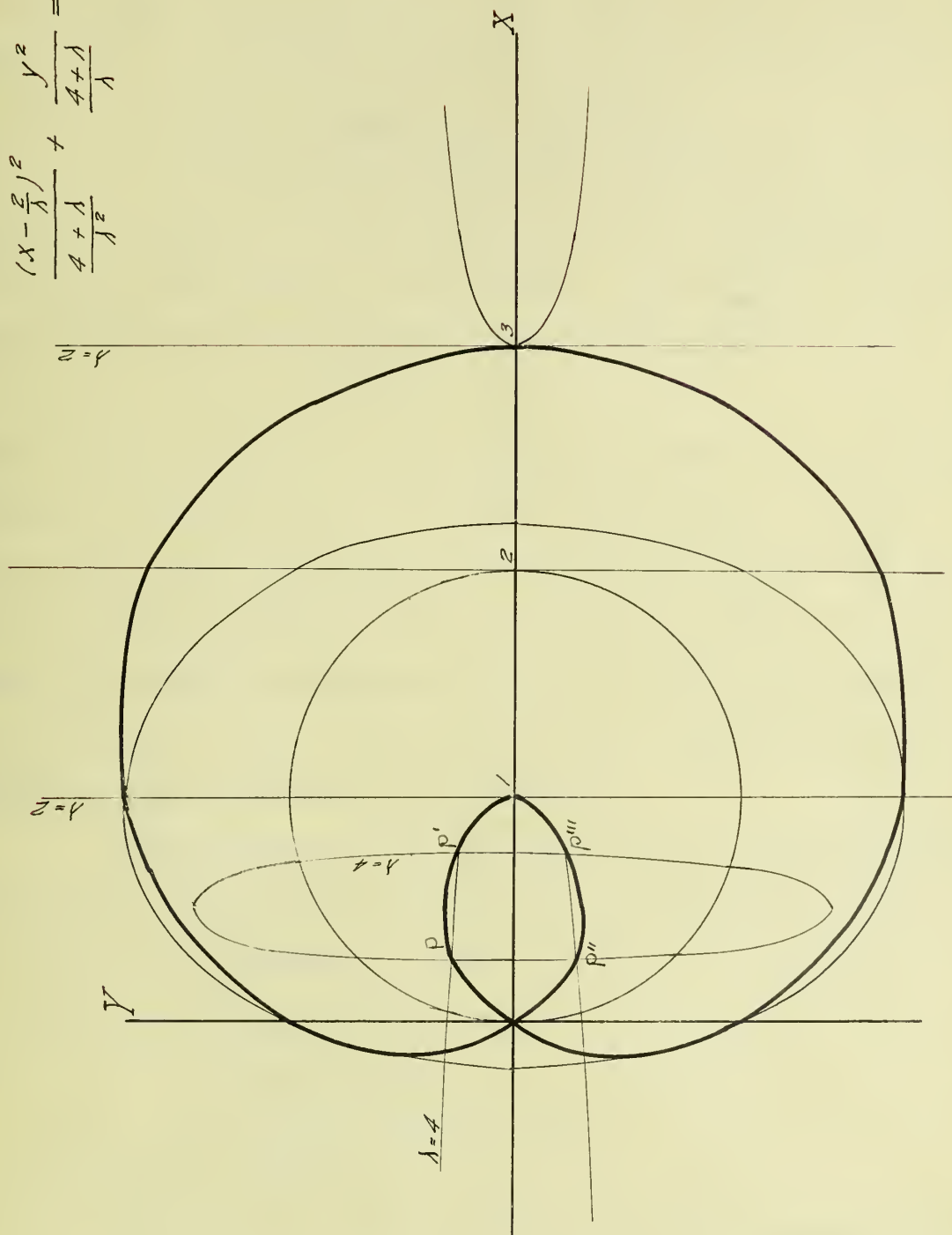


Fig. VII.

positive values of λ in the second pencil the center of the ellipses approach the origin, the vertical axes approach unity while the horizontal approaches zero. If the product of the two pencils is formed we have after simplifying,

$$x^4 + y^4 + 2x^2 + 2xy^2 - 4x^3 - 4xy^2 - y^2 = 0 \text{-----} (17)$$

Putting $y=mx$ in (17) we find there is a double point at the origin and the slopes of the tangents are $\pm \sqrt{\frac{3}{5}}$. If we reduce (17) to the polar form we have after simplifying

$$r = 2 \cos \theta \pm 1.$$

This equation is that of the Limaçon of Pascal. It is shown, Figure VI how the locus of (17) is constructed. In (16) let $\lambda = 4$, then the first is an hyperbola as indicated in Figure VI. The second for $\lambda = 4$ is an ellipse as indicated in figure. These two curves intersect at P, P', P'', P''' which are points on the locus of (17). In a similar way other points can be constructed as indicated in the figure.

As in example 8, we can show how (17) can be produced by two projective pencils of circles. Writing (17) in the form

$$(x^2 + y^2)(x^2 + y^2 - 4x + 3) - 4y^2 = 0, \text{-----} (17a)$$

it is at once apparent that

$$x^2 + y^2 + 2\lambda y = 0,$$

$$2y + \lambda(x^2 + y^2 - 4x + 3) = 0,$$

are the pencils required.

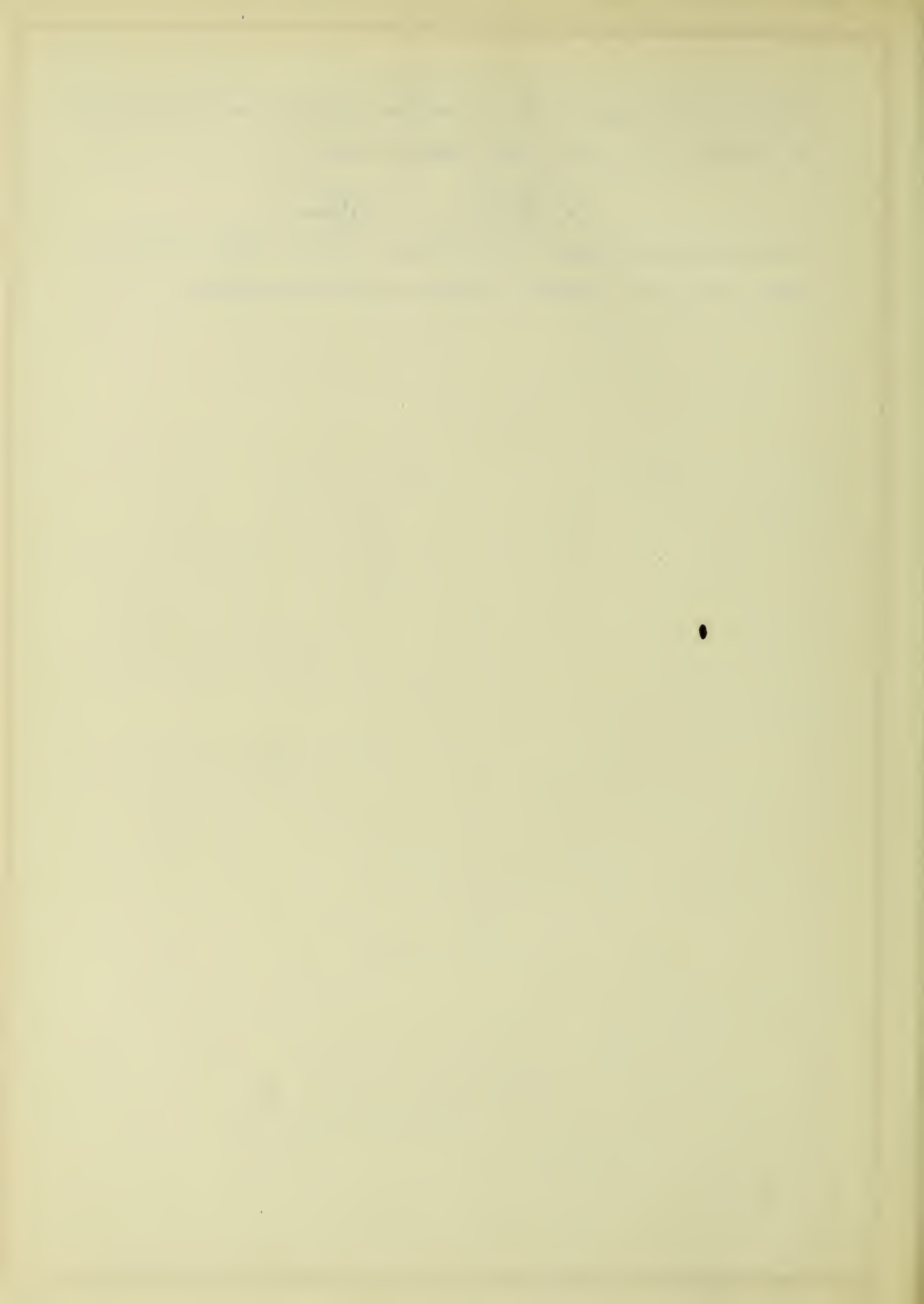
By the same method as was employed in example (8) it can be shown that the curve passes through the circular points at infinity. The circular points are also double points. For if we apply the transformation

$$x = \frac{x'}{x'-1}, \quad y = \frac{y'}{x'-1}$$

to (17a), and then form the partial derivatives with respect to x and y , it is at once observed that

$$I, \quad \begin{cases} x = 1 \\ y = i \end{cases} \quad II, \quad \begin{cases} x = 1 \\ y = -i \end{cases}$$

are solution of these differntial equations when equated to zero. Hence the circular points are double points.



Construction of Another Special Quartic from Geometric Conditions.

Another way of constructing a quartic which is of theoretical importance is by the aid of Pascal's Theorem. Before exhibiting this method we shall state and prove a few theorems with respect to passing a conic through the vertices and orthocenter of a triangle.

Theorem I.

If two conics described through four points are rectangular hyperbolas, all conics through these points are rectangular hyperbolas.

For the conics determine an involution on the line at infinity. By hypothesis two of the conics are rectangular hyperbolas, hence two pairs of the conjugate points on this line are determined by orthogonal rays. Therefore all pairs of conjugate points are determined by orthogonal rays and all conics through the four points have their asymptotes at right angles, and are rectangular hyperbolas.

Theorem II.

All conics through the vertices of a triangle and its orthocenter are rectangular hyperbolas.

For any side and a perpendicular from the opposite vertex constitute a degenerate rectangular hyperbola. Therefore there are two rectangular hyperbolas through the four points. Hence by theorem I all conics through these points are rectangular hyperbolas.

In the example that follows we shall use an equilateral triangle. The vertices and orthocenter of which gives four points.

The orthocenter will be the origin of a system of rectangular coordinates in which the x axis coincides with the altitude.

Since we must have five points, or four points and a line tangent to the conic, in order to construct other points, we shall take with the four points of the triangle a pencil of lines making equal angles with each other and also passing through the origin. By taking the four points with each line a conic is determined, and by the aid of Pascal's Theorem we can construct an indefinite number of points. All of these conics will be equilateral hyperbolas by the two theorems stated above.

Next, we must fix up another pencil corresponding to the hyperbolas. We shall take a pencil of concentric circles with the origin as center. In order for these circles to correspond with the hyperbolas we shall take a line parallel to the x axis at a convenient distance below the origin. This line will cut the tangents to the hyperbolas at points which we shall denote by $\lambda_0, \lambda_1, \lambda_2, \dots$ and which we may consider as parameters of the pencil τ of tangents at O to the pencil of hyperbolas. In order that the pencil of concentric circles shall be projective to the pencil of hyperbolas it is sufficient to make the pencil of polars π of any point, say A , with respect to the pencil of concentric circles. Let the inscribed circle of the triangle $A B C$ with its polar cutting the x axis at K correspond to the hyperbola through C and A , and $\lambda = \frac{1}{4}$, so that we have in the pencil of hyperbolas

$$(x^2 - x - y^2) - 2\lambda (xy + \frac{1}{2}y) \text{-----} (1)$$

for the projective pencil τ of tangents

$$\frac{y}{x} = \frac{-1}{\lambda} \text{-----}(2)$$

The intercepts of this pencil on the line $y + 1 = 0$ is evidently $x = \lambda$.

For the projective pencil of circles we can write

$$x^2 + y^2 = \lambda \text{-----}(3)$$

The pencil of polars of this with respect to the point $A(1,0)$ becomes

$$x = \lambda.$$

To construct the hyperbolas whose tangent cuts off on $y + 1 = 0$ the intercept λ , denote O by the symbol DE. Now according to Pascal's theorem construct any number of points of the hyperbola thus determined, make use of the scheme

$$\begin{array}{ccc} \begin{array}{c} A \ B) \\ D \ E) \end{array} \alpha, & \begin{array}{c} B \ C) \\ E \ F) \end{array} \beta, & \begin{array}{c} C \ D) \\ F \ A) \end{array} \gamma. \end{array}$$

In this α is determined, hence draw any Pascal line p through it and determine F belonging to this p .

See Figure VIII.

To construct the projective pencil of concentric circles with the pencil of polars $x = \lambda$, draw this polar p_λ , Figure IX, and find its intersection, P with the circle over OA as a diameter. Then OP is the radius of the required circle C_λ . Figure IX also shows the value of OP when λ is less than one.

In Figure VIII which shows the construction of the quartic

$$x^2 - x - y^2 - 2(x^2 + y^2)(xy + \frac{1}{2}y) = 0, \text{-----}(4)$$

λ is first equal to 3 and the corresponding circle intersects the hyperbola at the points P . These points are on the quartic.

Next for $\lambda = 5$ we get the points Q . These are also points on the quartic.

$$x^2 + y^2 = 1$$

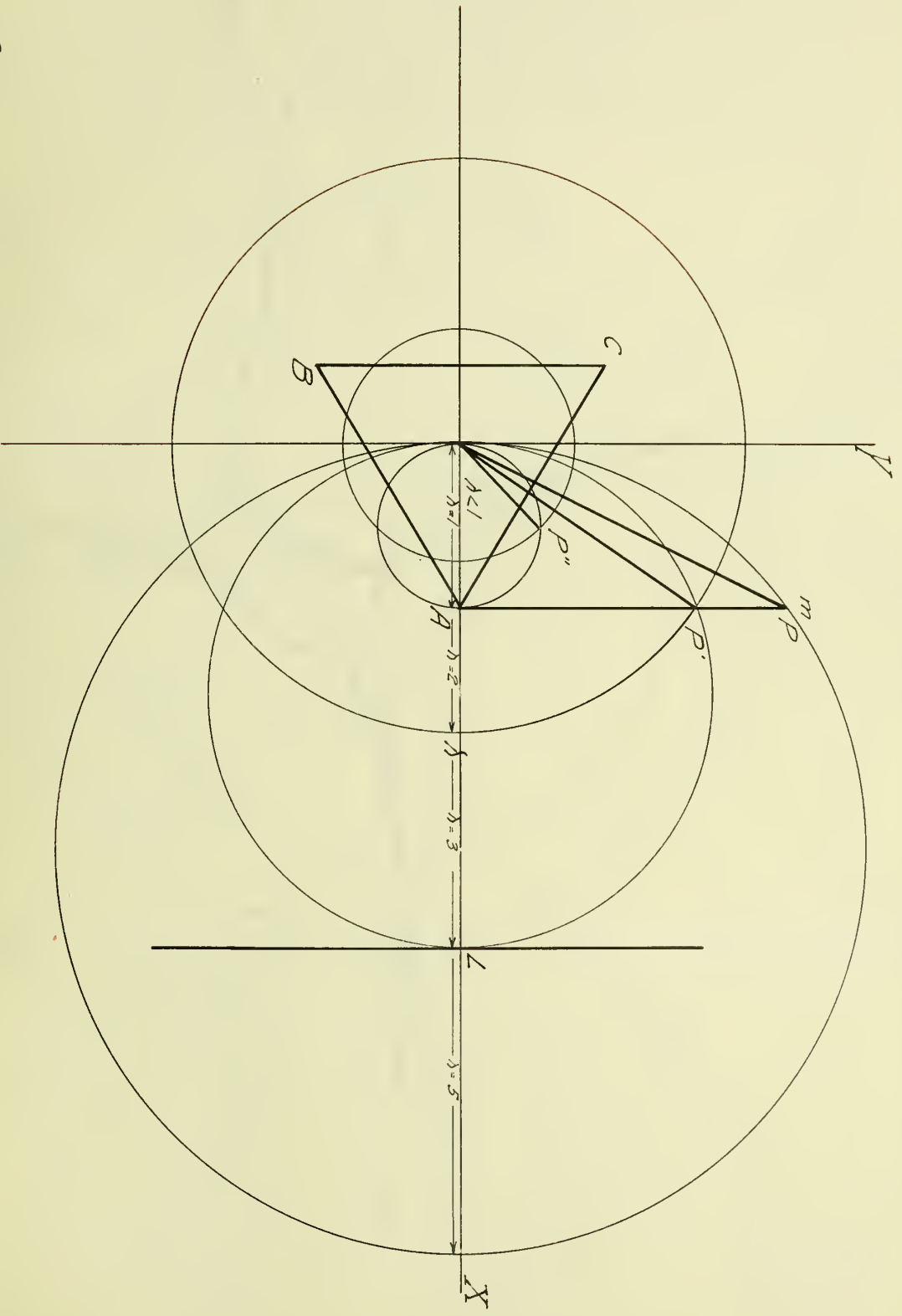
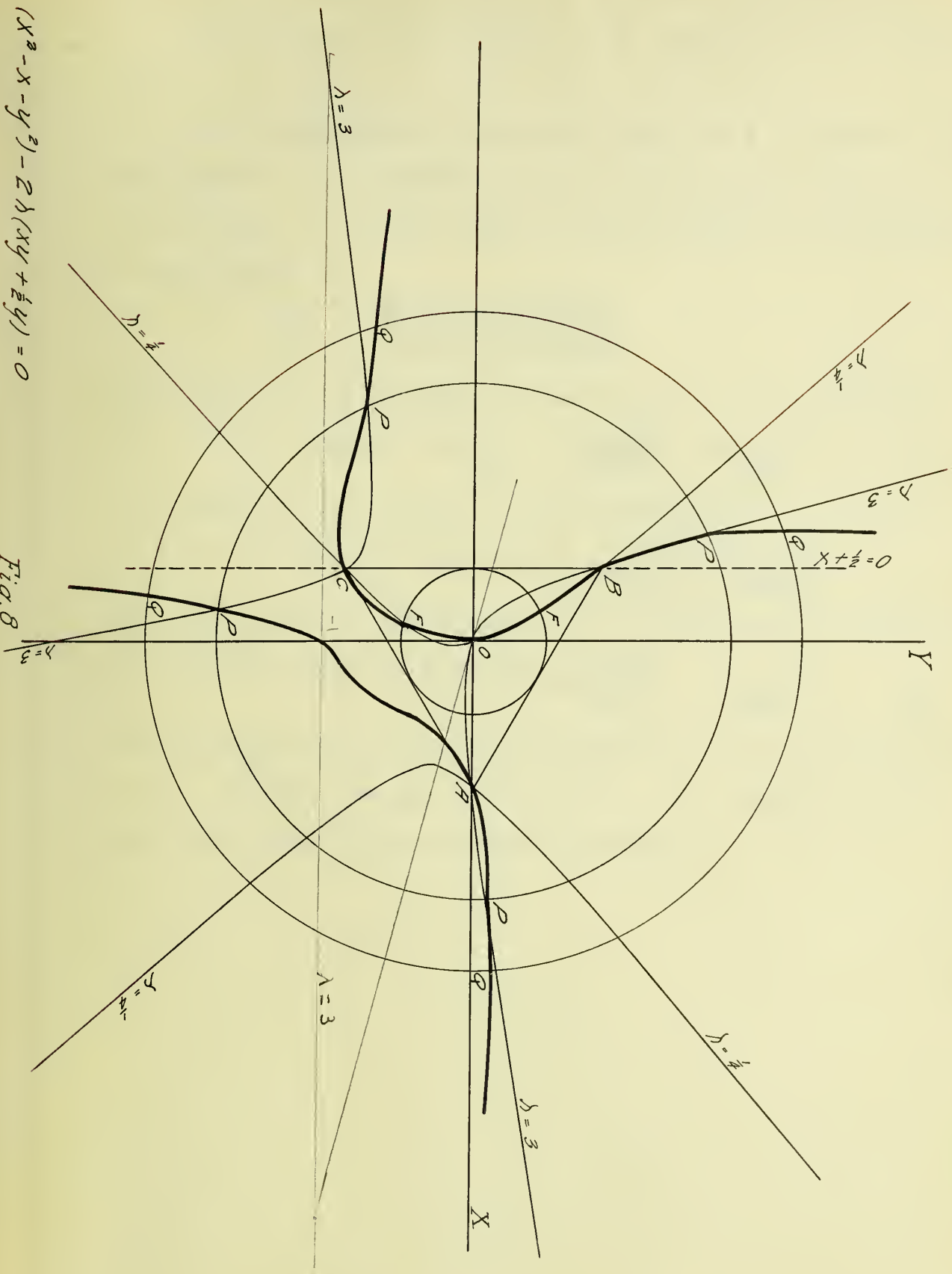


Fig. 19

$$(x^2 - x - y^2) - 2\lambda(xy + \frac{1}{2}y) = 0$$

Fig. 8



The triangle A B C in both figures is equilateral.

The heavy curve in Figure VIII represents the locus of (4).

The asymptotes of the quartic are $x + \frac{1}{2} = 0$ and $y = 0$. The slopes of the tangents at the vertices and orthocenter of the triangle A B C are found by substituting the coordinates of each point in

$$\frac{dy}{dx} = \frac{2x-1-2xy-2y^3-6x^2y}{2x^3+6xy^2+x^2+3y^2+2y}$$

The slopes of the tangents at the A, B, C, O and the point (0, -1) are respectively $\frac{1}{3}$, $-\frac{(2+\sqrt{3})}{\frac{1}{2}\sqrt{3}}$, $\frac{2-\sqrt{3}}{\sqrt{3}}$, ∞ , 1.

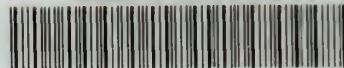
The slopes of the tangents to the hyperbolas at the points A, B, C, and O are, for $\lambda = \frac{1}{4}$, $\frac{4}{3}$, $\frac{8+\sqrt{3}}{-4\sqrt{3}}$, $\frac{8-\sqrt{3}}{4\sqrt{3}}$, -4.

Also the slopes of the tangents to the hyperbola at the points A, B, C and O when $\lambda = 3$ are $\frac{1}{9}$, $\frac{2+3\sqrt{3}}{-\sqrt{3}}$, $\frac{2-3\sqrt{3}}{\sqrt{3}}$, $-\frac{1}{3}$.

The corresponding points on the quartic for $\lambda = \frac{1}{4}$ are denoted by F. For $\lambda = 3$ we get the points P. In a similar way we find the points Q on the quartic for $\lambda = 5$.



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